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Dynamic programming approach to a Fermat type principle for heat flow

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Abstract

We consider nonlinear heat conduction satisfying a variational principle of Fermat type in the case of stationary heat flow. We review origins of a physical theory and transform it into a formalism consistent with irreversible thermodynamics, where the theory emerges as a consequence of the theorem of minimum entropy production. Applications of functional equations and the Hamilton–Bellman–Jacobi equation are effective when Bellman's method of dynamic programming is applied to propagation of thermal rays. Potential functions describing minimum resistance are obtained by analytical and numerical methods. For the latter, approximation schemes are developed. Differences between propagation of thermal and optical rays are discussed and it is shown that while simplest optical rays can be described by Riemmanian geometry, it is rather Finslerian geometry that is valid for thermal rays. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Consider a steady-state heat conduction in a rigid solid. When a thermal field is imposed by fixing the thermal gradient, the flow of thermal energy can be described in terms of 'thermal rays', the paths of heat flow determined by the direction of the temperature gradient and nonlinear properties of the conducting medium. When the thermal conductivity changes along the length of a thermal ray, the path along which the ray moves is, in general, curvilinear. Our purpose is prediction of the shapes of thermal rays, regardless of whether their curvilinearity is caused by the thermal inhomogeneity or material inhomogeneity of the medium. Here the thermal rays are shown to travel along paths satisfying the principle of minimum of entropy production which looks at first glance quite different from the well-known Fermat principle of minimum time (minimum optical length) for optical rays. However, taking into account that the minimum of entropy production is associated with the minimum resistivity of the path, it is easy to conclude that the minimum resistivity causes (in the dual problem) the maximum of heat flux through the medium or makes the residence time of heat in the medium as short as possible. This makes the principle for travel of thermal rays quite similar to that for propagation of light [6]. Our purpose is to investigate these phenomena by the method of dynamic programming [1] showing the similarities and differences between the optical and thermal phenomena.

2. Thermodynamics and propagation of steady thermal rays

Consider the entropy production functional describing the transfer of pure heat in a rigid solid under the assumption that the thermodynamic Hamiltonian vanishes, i.e. the rate dependent dissipation function Φ

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Nomenclature

A	variable area perpendicular to heat flow	$R(x, y)$	minimum resistance potential
A_0	constant area of transfer projected on axis y	S_σ	entropy generated during a finite period of time
c	bending constant for a thermal ray	T	temperature
H	Hamiltonian function	t	time
I	heat current through the area A	W^N	optimized performance function
k	Onsager's conductivity related to gradient of T^{-1}	ρ	thermal resistance as reciprocal of Onsager's conductance k
l	length parameter	x	direction perpendicular to the resistivity gradient
n	refraction coefficient	y	direction tangent to the resistivity gradient
p	momentum type integral, $\partial R/\partial y$		
R	total resistance of thermal path		

equals the state dependent dissipation function Ψ. This condition is associated with the requirement that an entropylike function generated along the kinetic paths is the true thermodynamic entropy which does not explicitly contain the time t

$$S_\sigma = \int_{t_1,\, v}^{t_2} L_\sigma \,\mathrm{d}V\,\mathrm{d}t \equiv \int_{t_1,\, v}^{t_2} (\Phi_s + \Psi_s)\,\mathrm{d}V\,\mathrm{d}t$$

$$= \int_{t_1,\, v}^{t_2} 2\Phi_s \,\mathrm{d}V\,\mathrm{d}t = \int_{t_1,\, v}^{t_2} \rho J_q^{\,2}\,\mathrm{d}V\,\mathrm{d}t, \tag{1}$$

where $\Phi_s = \Psi_s$ [5]. The symbol ρ designates the reciprocal of the well-known Onsager's coefficient k for the heat conduction. This reciprocal has the meaning of the specific resistance for heat transfer, hence its designation. The energy is transferred along the length dl by the cross-section perpendicular to the heat flux. The perpendicular crosssection has the area A which may change with l; the volume differential $\mathrm{d}V = A\,\mathrm{d}l$. As distinguished from more standard treatments, we integrate here over the volume V 'moving with the energy'; in this case, x and y are special Lagrange coordinates and the heat flow is attributed to motion of the same portion of energy rather than to flow through a fixed area in the space. We introduce the heat current $I = \mathrm{d}Q/\mathrm{d}t$ as the amount of the thermal energy received by the system per unit time. The heat Q is positive when it is added to the system. Then the heat flux densities satisfy $J_q = \mathrm{d}Q/A\mathrm{d}t$ or $J_q = I/A$, hence

$$S_\sigma = \int_{t_1, v}^{t_2} \rho\left(\frac{\mathrm{d}Q}{A\mathrm{d}t}\right)^2 \mathrm{d}V\,\mathrm{d}t = \int_{t_1,\, l}^{t_2} (\rho \mathrm{d}l)\left(\frac{\mathrm{d}Q}{A\mathrm{d}t}\right)^2 A\,\mathrm{d}t$$

$$= \int_{t_1,\, l}^{t_2} \left(\frac{\rho}{A}\,\mathrm{d}l\right)\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2 \mathrm{d}t. \tag{2}$$

As ρ has the meaning of the specific thermal resistance, the differential expression

$$\mathrm{d}R \equiv \rho\frac{\mathrm{d}l}{A} \tag{3}$$

defines the first differential of the total resistance R. The total resistance itself is the path integral

$$R \equiv \int_{l_1}^{l_2} \frac{\rho}{A}\,\mathrm{d}l \tag{4}$$

The quantity R increases with the total length l and decreases with the cross-sectional area A. With this definition, Eq. (2) can be transformed in to a popular form that describes the generation of the Joule heat within a conductor. Indeed, as shown by Eq. (5) below, in the frame of the variable Q the entropy production is

$$S_\sigma = \int_{l_1}^{l_2} \left(\frac{\rho}{A}\right) \mathrm{d}l \int_{t_1}^{t_2} \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2 \mathrm{d}t = \int_{Q_1}^{Q_2} RI\,\mathrm{d}Q$$

$$= \int_{Q_1}^{Q_2} \left(T_2^{-1} - T_1^{-1}\right)\mathrm{d}Q = \int_{t_1}^{t_2} RI^2\,\mathrm{d}t \tag{5}$$

This shows that the difference of thermal potentials between the two subsystems 1 and 2, $\Delta T^{-1} = 1/T_2 - 1/T_1$, causes the flow of the thermal energy $\mathrm{d}Q \equiv \mathrm{d}Q_2$ along the total resistance R to heat the subsystem 2. The Ohm's law for heat conduction holds in the form

$$I \equiv -\frac{\mathrm{d}Q_1}{\mathrm{d}t} = \frac{\mathrm{d}Q_2}{\mathrm{d}t} = \frac{\Delta T^{-1}}{R} = \frac{T_2^{-1} - T_1^{-1}}{R} \tag{6}$$

At the steady state

$$S_\sigma = \int_{Q_1}^{Q_2} \left(T_2^{-1} - T_1^{-1}\right)\mathrm{d}Q = \int_{t_1}^{t_2} RI^2\,\mathrm{d}t$$

$$= \left(T_2^{-1} - T_1^{-1}\right)Q \tag{7}$$

Note that for an unsteady state process, when two bodies exchange heat and the system is isolated as the whole, the correct result is one half of the expression given in Eq. (7). This is because in the unsteady state process the heat current I is not constant in time but decreases gradually to zero. Here, however, we are interested in the steady-state behavior for which the constancy of I along the path holds.

On the other hand, another transformation of Eq. (2) shows that the entropy production during the time $\Delta t = t_2 - t_1$ can be expressed in the form

$$S_\sigma = \int_{t_1}^{t_2}\int_{t_1}^{t_2}\left(\rho\frac{\mathrm{d}l}{A}\right)\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2 \mathrm{d}t$$

$$= (t_2 - t_1)\int_{t_1}^{t_2}\left(\rho I^2 A^{-1}\right)\mathrm{d}l, \tag{8}$$

Thus, the steady intensity of the entropy generation or S_σ per unit time can be written as

$$P_\sigma = \frac{S_\sigma}{t_2 - t_1} = \int_{l_1}^{l_2}\left(\rho I^2 A^{-1}\right)\mathrm{d}l = \int_{l_1}^{l_2}\Pi\,\mathrm{d}l, \tag{9}$$

where, by definition,

$$\Pi \equiv \rho I^2 A^{-1}, \tag{10}$$

plays formally the role of a momentum type quantity. Even usual units of momentum (kgm/s) can be assigned to Π, after multiplying it by a suitable constant. This can be done by noting that the power of the entropy produced P_σ in Eq. (9) can be expressed in units of the entropy itself by multiplying this equation by a time constant $\tau' = \mathrm{h/mc^2}$, where h is Planck's constant, c is light speed and m is the mass of the molecule in the solid system

$$P_\sigma = \frac{S_\sigma h}{\Delta t m c^2} = \int_{l_1}^{l_2}\left(\frac{\rho I^2 h}{A m c^2}\right)\mathrm{d}l = \int_{l_1}^{l_2}\frac{\Pi h}{m c^2}\,\mathrm{d}l. \tag{11}$$

This quantity, in turn, can be expressed in units of action, after multiplying it by h/k_B, where k_B is the Boltzmann constant. Consequently, the thermal action $(h/k_\mathrm{B})P_\sigma$ takes the form of an integral representative to variational principles of Mapertuis–Fermat type

$$A = \frac{S_\sigma h^2}{\Delta t m c^2 k_\mathrm{B}} = \int_{l_1}^{l_2}\left(\frac{\rho I^2 h^2}{A m c^2 k_\mathrm{B}}\right)\mathrm{d}l = \int_{l_1}^{l_2} p\,\mathrm{d}l, \tag{12}$$

where

$$p \equiv \frac{\Pi h^2}{m c^2 k_\mathrm{B}} = \frac{\rho I^2 h^2}{A m c^2 k_\mathrm{B}} \tag{13}$$

is the 'thermal momentum' in the usual units, kg m/s. We stress that A is the area of the crosssection perpendicular to the flow. The variational principle is

$$\delta\int_{l_1}^{l_2}\left(\frac{\rho I^2 h^2}{A m c^2 k_\mathrm{B}}\right)\mathrm{d}l = \delta\int_{l_1}^{l_2} p\,\mathrm{d}l = 0. \tag{14}$$

Certain specific coordinates x and y are usually applied to describe the problem; in this case

$$\delta\int_{l_1}^{l_2} p\,\mathrm{d}l = \delta\int_{l_1}^{l_2} p(x, y, u)\left(\sqrt{1+u^2}\right)\mathrm{d}x = 0, \tag{15}$$

where $u = \mathrm{d}y/\mathrm{d}x$ is the slope of the tangent to the path. In this reference frame the local resistivity of heat flow changes along the axis x, the axis y is tangent to a surface of constant resistivity $\rho = C$ and $u = \mathrm{d}y/\mathrm{d}x$ is the local direction of the gradient of temperature reciprocal T^{-1}.

3. Mechanical and optical analogies

It is essential that various specific forms of the function $p(x, y, u)$ may refer to diverse physical phenomena. For mechanical motions with $p = mv$ and the energy conservation in the form $(1/2)mv^2 + V(x, y) = h$, one obtains $p = [2m(h - V(x, y))]^{1/2}$; in this case p is independent of u and the variational equation can be rewritten in the form

$$\delta\int_{l_1}^{l_2} p\,\mathrm{d}l = \delta\int_{x_1}^{x_2}\sqrt{2m\left(h - V(x, y)\right)\left(1+u^2\right)}\,\mathrm{d}x$$

$$= 0. \tag{16}$$

In the simplest (homogeneous) case of optical phenomena p may also be independent of u. However, in thermal phenomena p, Eq. (13), includes the perpendicular cross-section A, thus p may inherently change with $u = \mathrm{d}y/\mathrm{d}x$ depending on how A varies along the path. We shall see this later as we now shall focus on the light propagation, where the transition time between two points may be described by an (inhomogeneous) functional

$$\tau \equiv t - t_0 = \int_{x_0}^{x} v(y, x)^{-1}\sqrt{1+u^2}\,\mathrm{d}x, \tag{17}$$

which contains the actual propagation speed v in the denominator. When this equation is multiplied by the light speed in vacuum, c_0, it takes the form expressing the optical length X

$$X = c_0\tau \equiv c_0(t - t_0) = \int_{x_0}^{x} n(y, x)\sqrt{1+u^2}\,\mathrm{d}x, \tag{18}$$

where $n = c_0/v$ is the refraction coefficient. Due to the obvious analogy between Eqs. (16) and (18), one can restrict further analysis of the mechanical case and the homogeneous optical case to the latter function. The Euler–Lagrange equation is

$$\sqrt{1+u^2}\frac{\partial n}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{nu}{\sqrt{1+u^2}}\right\} = 0. \tag{19}$$

This equation admits a simple and important interpretation. If the tangent to the extremal makes the angle α with the axis x, then

$$\frac{dy}{dl} = \frac{dy}{\sqrt{dx^2 + dy^2}} = \frac{u}{\sqrt{1+u^2}} = \sin\alpha. \tag{20}$$

Eq. (20) is then reduced to

$$\frac{\partial n}{\partial y} - \frac{d}{dl}(n\sin\alpha) = 0. \tag{21}$$

This equation describes a geometrical property of the extremal path, which is independent of the choice of axes. Therefore, its interpretation can use any convenient system of axes. As n depends on x and y only, the equation n = constant is that of the plane curve and by varying the constant we obtain a family of the curves, the so-called level curves. If O is point on the extremal, let us take it as the origin and the normal and tangent to the level curve through O as the x and y axes respectively. Now at O the tangent to the curve n = constant is perpendicular to the axis x, hence $(\partial n/\partial x)/(\partial n/\partial y)$ must be infinite and so $\partial n/\partial y = 0$. Eq. (21) then becomes the Snell (sine) law of refraction for continuous systems

$$n\sin\alpha = \text{constant} \tag{22}$$

which holds for all points along the extremal. This equation is not restricted to the optical case. It is easy to see that in the mechanical case the level curves are given by $(h-V)^{1/2} = \text{constant}$, and, as this is equivalent to V = constant, the level curves are curves of the constant potential energy. The condition $(h-V)^{1/2}\sin\alpha = \text{constant}$ holds in this case, where $(1/2)\Pi - \alpha$ is the angle at which tangents to two curves, the level curve and the extremal, intersect.

4. Boundary conditions and tangent law of bending for thermal rays

Consider now which of the above properties cease to be valid in the case of thermal rays. The integration is made along the trajectory of the heat flow, a 'thermal ray'. Because the thermal energy in a rigid solid is a conserved quantity, the energy flux I is constant in a steady state heat conduction. In this case, as shown above by Eqs. (9) or (14), the general principle of the entropy production has the special form of a variational principle of Mapertuis–Fermat type. As it is well-known from analytical mechanics, a principle of least action type must hold in this case and an energylike quantity must be constant along the ray. In the considered case this requirement is satisfied by constancy of the thermodynamic Hamiltonian, Φ–Ψ.

Since the units of the entropy produced S_σ are J K^{-1}, the units of the quantity p are J K^{-1} m^{-1}, consistent with units of the specific resistance ρ, K^{-1} J^{-1} ms, resulting from the heat conduction law in Onsager's form, $\rho J = \text{grad}\ T^{-1}$. Correspondingly, the total resistance R has units K^{-1} J^{-1} s in agreement with Eq. (3). Thus the product RI has units K^{-1}, consistent with the fact that this product represents the temperature reciprocal in agreement with Ohm's law for heat, Eq. (6). The product $\rho dQ/dA$ has units K^{-1} m^{-1} s.

A variational principle of Mapertuis–Fermat type was constructed by Keizer [5] for Onsager's model in the abstract configuration space of the variables α, which are even with respect of the time reversal. Here, however, the variational principle works in the physical space x, y, z of our real word, thus the problem is physically more interesting than that in the abstract space.

Here we consider heat flux as "rays", for which a sort of refraction law can be formulated in an inhomogeneous medium in which the thermal conductivity changes with position. The appropriate way to show that a sort of refraction law exists was found by Tan and Holland [13] who have shown the essential role of the boundary conditions for heat flow through a discontinuous surface at which the heat conductivity has a jump, and derived the so-called tangent law of refraction. In particular, the surface of discontinuity may be an interface separating two phases; the Onsager's conductivity in the first phase is k_1 and that in the second phase is k_2. The tangent law is different from Snell's law of refraction for waves, with the tangents of the angles of incidence and refraction replacing the sines and the reciprocal of the Onsager's conductivity taking the place of the refractive index. The tangent law is known for the electric field intensity at the boundary between two dielectrics; again the tangents of the angles of incidence and refraction replace the sines in Snell's law. The same tangent law of refraction should also apply to potential fields in general.

When the number of phases (the stages with constant heat conductivity) tends to infinity, a continuous problem can be stated. We conveniently assume that the heat conductivity is a function of x only; then the axis x is perpendicular to the parallel surfaces of constant conductivity. The assumption of parallel surfaces of constant k is not really restrictive, because over a small enough area any smooth surface can be regarded as locally flat. When the original system is now rotated about the axis x until the thermal ray starts in the x–y plane, then the physics does not change, that is, it must, by symmetry remain in this plane. This means that a two-dimensional analysis is allowed. The heat rays travel along the direction of the negative gradient of temperature, perpendicular to the isothermal surfaces. Our reference frame is such that the vector of the temperature gradient lies in the plane x–y and the surfaces of constant k are represented by the lines $x =$ constant.

Let us describe several cases in which tangent law boundary conditions hold. First, the boundary conditions at the surface separating two isotropic dielectrics must be consistent with the condition rot $\mathbf{E} = 0$ which requires, for an isotropic homogeneous surface, the continuity of the tangential component, $E_{1t} = E_{2t}$ (indices 1 and 2 refer to phase 1 and phase 2). Second, the boundary conditions must be consistent with the condition div $\mathbf{D} = 0$ or div$(\varepsilon\mathbf{E}) = 0$; this requires the continuity of the normal component of $\mathbf{D}$ otherwise the divergence could not vanish. Thus the condition is $\varepsilon_1\mathbf{E}_{1n} = \varepsilon_2\mathbf{E}_{2n}$. Analogous boundary conditions hold for electric conductors. The analogy follows from the fact that, for conductors, the first condition, rot $\mathbf{E} = 0$, holds same for dielectrics, whereas the second condition div$(\sigma\mathbf{E}) = 0$, differs from its static counterpart only by the presence of the electric conductivity σ replacing the dielectric constant. Thus, for electric conductors, the second boundary condition is $\sigma_1 E_{1n} = \sigma_2 E_{2n}$.

Quite generally, for potential flows, the tangent component of the intensity (**E**, grad P, grad T, etc.) is continuous across the interface, $E_{1t} = E_{2t}$. The normal component of the same intensity may have a jump resulting from different transport conductivities, $\sigma_1 E_{1n} = \sigma_2 E_{2n}$. On the other hand, these boundary relations can be expressed in terms of the conserved fluxes **j**; in this representation the tangent components have jumps, $j_{t1}/\sigma_1 = j_{t2}/\sigma_2$, and the normal components are continuous, $j_{n1} = j_{n2}$. Also, we can express these conditions in terms of the potential ϕ such that $\mathbf{E} = \text{grad}\,\phi$; in this representation, for a homogeneous interface the continuity condition for the tangent component of **E** is equivalent to the continuity of ϕ itself; thus

$$\phi_1 = \phi_2, \tag{23}$$

$$\sigma_1 \frac{\partial \phi_1}{\partial n} = \sigma_2 \frac{\partial \phi_2}{\partial n}. \tag{24}$$

Following Tan and Holland [13], but working in the entropy representation as the only correct one for the energy transfer, we shall derive the tangent law of refraction for heat conduction from the boundary conditions of the temperature continuity and the energy conservation at the interface. We assume the steady state heat conduction through a smooth interface without interfacial thermal resistance and with no heat sources or sinks. Under these conditions the temperature will be continuous across the interface and no question of reflection arises because the ray direction at each point is uniquely determined by that of the temperature gradient. In Fig. 1, α_1 and α_2 are respectively the angles of incidence and refraction, whereas k_1 and k_2 are the Onsagerian conductivities of the two media. For an interface characterized by good thermal contact between the media, that is, by temperature continuity across the interface, the boundary conditions can be written as

$$|\nabla T^{-1}|_1 \sin\alpha_1 = |\nabla T^{-1}|_2 \sin\alpha_2 \tag{25}$$

for the tangent component and

$$k_1|\nabla T^{-1}|_1 \cos\alpha_1 = k_2|\nabla T^{-1}|_2 \cos\alpha_2 \tag{26}$$

for the normal component in absence of sources and sinks of heat. The law of the refraction for heat conduction rays follows as a ratio of these equations

$$\rho_1 \tan\alpha_1 = \rho_2 \tan\alpha_2 \tag{27}$$

where $\rho = 1/k$ is the Onsagerian thermal resistivity. This is analogous to Snell's law with the tangent replacing the sine and the thermal resistivity replacing the refractive index; the law was originally formulated with the reciprocal of usual thermal conductivity λ [13] which we, however, attribute here to the reciprocal of Onsagerian conductivity k as the more proper quantity than λ because of the basic link of the former with the entropy production.

Examining the deviation of the refracted ray from the incident ray, it is convenient to introduce the relative thermal resistivity of the second medium with respect to the first; $\beta = \rho_2/\rho_1$. In terms of β and the angle of incidence, α_1, the angle of refraction is

$$\alpha_2 = \arctan(\tan\alpha_1/\beta) \tag{28}$$

whereas the angle of deviation of the reflected ray, $\Delta = \alpha_2 - \alpha_1$ satisfies the equation

$$\Delta \equiv \alpha_2 - \alpha_1 = \arctan(\tan\alpha_1/\beta) - \alpha_1. \tag{29}$$

Tan and Holland [13] obtained figures showing the angles of refraction, α_2, and deviation, Δ, as functions of the incidence angle α_1 and the resistivity ratio, $\beta = \rho_2/\rho_1$. The case $\beta > 1$ is the optical analog of refraction from the rarer to the denser medium, and inversely. In contrast to Snell's refraction law in optics,

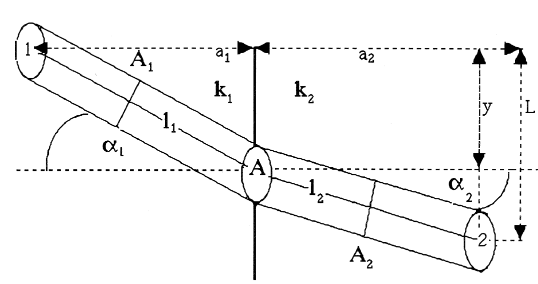

Fig. 1. Illustration of Fermat principle for heat conduction.

where the deviation Δ decreases monotonically with the incidence angle α_1, in the tangent law case the deviation attains an extremum and approaches zero for both grazing and normal incidences, $\alpha_1 = 0$ and $\alpha_1 = 90°$, respectively. In the limit as β tends to zero, the refraction angle α_2 approaches 90° which means that heat flux entering a perfect thermal conductor is parallel to the interface. Otherwise, as β tends to infinity, the refraction angle α_2 becomes zero, which means that heat flux entering a perfect thermal insulator must be perpendicular to the interface.

The specific resistance ρ, K^{-1} J^{-1} m s, resulting from the heat conduction law in Onsager's form, $\rho J = \text{grad}\, T^{-1}$ can be connected with the usual heat conductivity λ. Since the Fourier law must hold, $T^2 \rho J = -\text{grad}\, T$, the thermal resistance ρ used here can be evaluated on the basis of the thermal conductivity data; $\rho = T^{-2}\lambda^{-1}$. The corresponding total resistances: $R_1 = \rho_1 l_1/A_1$ and $R_2 = \rho_2 l_2/A_2$ (those of the entropy representation) defined in analogy with the electric resistance, have units K^{-1} J^{-1} s. Here A_1 and A_2 are the cross-sectional areas of a "tube" of the heat flux in the medium 1 and 2, respectively. The products $R_1 I$ and $R_2 I$ have units K^{-1}, consistent with the fact that these products represents the differences of the temperature reciprocals, ΔT_1^{-1} and ΔT_2^{-1}, in the two media.

We shall now show the essential role of the area perpendicular to the transferred heat ray on the tangent law of heat bending. In Fig. 2 the heat flux travels between two fixed points, 1 and 2. If A_0 is the constant area of a flux tube intercepted by the interface (the constant area of projection of the heat flux tube cross-sectional area on the surface of constant resistivity), then the cross-sectional areas of the flux tubes in the two media are

$$A_1 = A_0 \cos\alpha_1, \qquad A_2 = A_0 \cos\alpha_2. \tag{30}$$

Thus, the total thermal resistance between the points 1 and 2 is

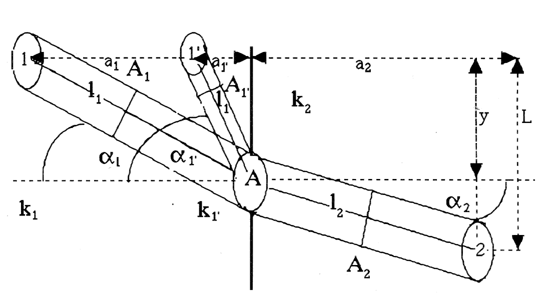

Fig. 2. Decrease in the area A_1 perpendicular to the transferred heat ray with the increasing difference between thermal conductivities k_1 and k_2, at constant k_2.

$$R_{1,2} = \frac{\rho_1 l_1}{A_1} + \frac{\rho_2 l_2}{A_2} = \frac{1}{A_2}\left(\frac{\rho_1 l_1}{\cos\alpha_1} + \frac{\rho_2 l_2}{\cos\alpha_2}\right). \tag{31}$$

Substituting the values of $\cos\alpha_1$ and $\cos\alpha_2$ in this equation from Fig. 1, as

$$\cos\alpha_1 = \frac{a_1}{\sqrt{a_1^2 + y^2}}, \quad \cos\alpha_2 = \frac{a_2}{\sqrt{a_2^2 + (L-y)^2}} \tag{32}$$

and the lengths l_1 and l_2 as

$$l_1 = \sqrt{a_1^2 + y^2}, \quad l_2 = \sqrt{a_2^2 + (L-y)^2} \tag{33}$$

one obtains

$$R_{1,2} = \frac{1}{A_0}\left(\frac{\rho_1\left(a_1^2 + y^2\right)}{a_1} + \frac{\rho_2\left(a_2^2 + (L-y)^2\right)}{a_2}\right). \tag{34}$$

We stress that it is the vertical coordinate y of the intersection point with the interface which is allowed to vary in our case. This is because the location of the intersection point of the thermal ray with the interface may occur for various vertical coordinates y while the horizontal coordinate x of that point is always constant and equal to a_1. Since $y/a_1 = \tan\alpha_1$ and $(L-y)/a_2 = \tan\alpha_2$, the condition requiring the first derivative $dR_{1,2}/dy$ to vanish

$$\frac{dR_{1,2}}{dy} = \frac{2}{A}\left(\frac{\rho_1 y}{a_1} - \frac{\rho_2(L-y)}{a_2}\right) = 0 \tag{35}$$

is equivalent with the requirement that the tangent law, Eq. (27), is satisfied. Differentiating Eq. (35) with respect to y once again, we obtain

$$\frac{d^2 R_{1,2}}{dy^2} = \frac{2}{A}\left(\frac{\rho_1}{a_1}\right) > 0 \tag{36}$$

which proves the minimum property of $R_{1,2}$ at the stationary point.

Consequently the postulate that the heat flux follows the trajectory of the least resistance is the correct physical principle that leads to the tangent law (27), the consequence of the boundary conditions (25) and (26). Along with the continuity of the normal component for a heat vector $\mathbf{H}$, satisfying $\text{div}\,\mathbf{H} = 0$, $\rho_e = \text{div}\,\mathbf{D}$, or — in the steady state — $\text{div}\,\mathbf{J} = 0$, the conditions implying the tangent law, Eqs. (25) and (26), may be seen as the consequence of the energy conservation and the principle of least resistance. As the latter is incorporated in the theorem of the least entropy production, we can regard Eqs. (25) and (26) as those stemming from the minimum entropy production applied to the conserved heat flux.

5. Continuous medium and tangent law of bending for thermal rays

Let us consider now the continuous medium with the variable thermal resistivity which is described by Eq. (14). We work again in the entropy representation. In our system, the surfaces of constant thermal resistivity are planes perpendicular to the axis x, i.e. the thermal resistivity is a continuous function of x only. Let us imagine that we rotate the system about the axis x until the gradient of the temperature reciprocal is parallel to the x–y plane. A set of flux tubes with flowing energy can be defined as in the discrete problem described above. While in the discrete problem in the entropy representation the total resistances are $R_1 = \rho_1 l_1/A_1$ and $R_2 = \rho_2 l_2/A_2$, in the continuous problem these relations are represented jointly by the single local relationship, $R = \rho A^{-1}\mathrm{d}l$, in which A is the variable cross-sectional areas of a "tube" of the heat flux perpendicular to the flow in a inhomogeneous medium. The products $R_1 I$ and $R_2 I$ have units K^{-1} and describe the differences in the temperature reciprocals.

Again we test the postulate that the path of the heat flowing between two fixed points 1 and 2 is that along which the total thermal resistance is minimum. As in the discrete problem, $A = A_0 \cos\alpha$, where A_0 is the constant (x-independent) area of the flux tubes intercepted by the interface. The variable cross-sectional area of the flux tube in the medium is described by a continuous counterpart of Eq. (30)

$$A = A_0 \cos\alpha \tag{37}$$

(see Fig. 2) whereas the incidence angle varies with x according to the formula

$$\cos\alpha = \frac{\mathrm{d}x}{\mathrm{d}l} = \frac{\mathrm{d}x}{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}} \tag{38}$$

In the above equations α is the angle between the thermal gradient (or the thermal ray) and a normal to the planes of constant resistivity. Eqs. (14), (37) and (38) then imply the formulation

$$\delta \int_{l_1}^{l_2} \frac{\rho}{A}\,\mathrm{d}l = \delta \int_{l_1}^{l_2} \frac{\rho(x)}{A_0 \cos\alpha}\,\mathrm{d}l = \delta \int_{x_1}^{x_2} \rho(x) \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{A_0\,\mathrm{d}x/\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}} = 0 \tag{39}$$

which describes the vanishing variation for the function of total resistance defined as

$$R_{1,2} \equiv \int_{t_1}^{t_2} \left(\frac{\rho(x)\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)}{A_0\,\mathrm{d}x^2} \right) \mathrm{d}x = \int_{x_1}^{x_2} A_0^{-1} \rho(x)\left(1 + (\mathrm{d}y/\mathrm{d}x)^2\right) \mathrm{d}x = 0. \tag{40}$$

A comparison of Eqs. (15) and (40) proves that in the thermal case the function $p(x, y, u)$ of Eq. (15) has the form

$$p(x, y, u) = C\rho(x)\sqrt{1 + u^2}, \tag{41}$$

where C is a constant. The dependence of p on u is caused by the change of the area A perpendicular to the flow with x, as shown in Fig. 2. It is Finslerian rather than Riemmanian geometry which describes well problems of this sort [4].

6. Continuous problem of bending as an optimal control problem

However, in this work, the thermal problem will be regarded as an optimal control problem for the minimum of the resistivity integral

$$\min \int_{x_1}^{x_2} A_0^{-1} \rho(x)\left(1 + u^2\right) \mathrm{d}x \tag{42}$$

subject to the control u defined by the simple state equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u. \tag{43}$$

Eq. (42) contains the Lagrangian type integrand, l_0, in which x is the independent variable and which does not contain explicitly the state variable y.

Now, according to a general procedure for variational problems [8,7,2] we consider the minimal resistance function

$$R\left(x^i, y^i, x^f, y^f\right) \equiv \min\left\{ \int_{x^i}^{x^f} l_0(x, y, u)\,\mathrm{d}x \right\} = \min \int_{x^i}^{x^f} A_0^{-1} \rho(x)\left(1 + u^2\right) \mathrm{d}x \tag{44}$$

in terms of final states and times. The Hamilton–Bellman–Jacobi equation (HJB equation) for the above functional follows in the form

$$\frac{\partial R}{\partial x} + \max_u \left\{ \frac{\partial R}{\partial y} u - A_0^{-1} \rho(x)\left(1 + u^2\right) \right\} = 0. \tag{45}$$

For the unconstrained control u, this is equivalent with the two equations

$$\frac{\partial R}{\partial x} + \frac{\partial R}{\partial y} u - A_0^{-1} \rho(x)\left(1 + u^2\right) = 0 \tag{46}$$

and

$$\frac{\partial R}{\partial y} - 2A_0^{-1} \rho(x) u = 0. \tag{47}$$

Substituting into the first equation for the derivative $\partial R/\partial y$ its value obtained from the second equation and then eliminating R from the two equations by comparison of the mixed second order derivatives yields the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\rho(x)\left(1+(\mathrm{d}y/\mathrm{d}x)^2\right)}{\partial(\mathrm{d}y/\mathrm{d}x)}=\frac{\partial\rho(x)\left(1+(\mathrm{d}y/\mathrm{d}x)^2\right)}{\partial y} \tag{48}$$

which, in view of the y-independent l_0, simplifies to the relationship

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2\rho(x)(\mathrm{d}y/\mathrm{d}x)\right)=0 \tag{49}$$

or, since $\mathrm{d}y/\mathrm{d}x=\tan\alpha$

$$\rho(x)\tan\alpha(x)=c, \tag{50}$$

where c is a constant. This is the tangent law of bending for an inhomogeneous medium in which the thermal resistivity is a function of x. Eq. (50) extends the discrete Eq. (27) to continuous systems.

Since the equation of an extremal is a second order differential equation, its solution depends on two integration constants. When the function $\rho(x)$ is known, variables in Eq. (49) can be separated. Integration between an initial point (x^0, y^0) and an arbitrary final point (x, y) yields the general integral

$$y=y^0+c\int_{x^0}^{x}\frac{\mathrm{d}x'}{\rho(x')}, \tag{51}$$

where the integration constants are c and y^0. Following Tan and Holland [13] we consider the case when the thermal resistivity increases exponentially with x

$$\rho(x)=\rho^0\exp(\gamma x). \tag{52}$$

When the solution between the points (x^0, y^0) and (x, y) is considered, Eq. (51) yields

$$y=y^0+\frac{c}{\rho^0}\int_{x^0}^{x}\frac{\mathrm{d}x'}{\exp(\gamma x')}=\frac{c}{\gamma\rho^0}\left(1-\exp(-\gamma x)\right) \tag{53}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{c}{\rho^0}\exp(-\gamma x). \tag{54}$$

In the above equations the ratio c/ρ^0 is the initial slope $(\mathrm{d}y/\mathrm{d}x)^0$ at the point (x^0, y^0). The latter equation shows that the slope of the heat ray decreases exponentially with x, thus turning toward the direction of the resistivity gradient. Indeed, in order to minimize the total resistance, the ray spanned between two given points must take the shape that assures that its relatively large part resides in the 'rarer' region of the medium. This is in agreement with the tangent law of refraction from a rarer to a denser medium. Eq. (53) proves that as x tends to infinity, y approaches the asymptotic value $c/(\gamma\rho^0)$. For an infinite ratio ρ^0/c, or for the vanishing initial slope $(\mathrm{d}y/\mathrm{d}x)^0$, we obtain $y=0$ and $\mathrm{d}y/\mathrm{d}x=0$ for all x. In this case a thermal ray initially in the direction of the resistivity gradient propagates undeviated. Otherwise, considering the inverted form of Eq. (53)

$$x=-\gamma^{-1}\ln\left(1-\frac{\gamma\rho^0 y}{c}\right) \tag{55}$$

and Eq. (54) we conclude that if $\rho^0 c=0$ then x and $\mathrm{d}x/\mathrm{d}y$ are zero for all y. This means that a ray perpendicular to the resistivity gradient (tangent to a surface of the constant resistivity) also propagates undeviated. This is consistent with the discrete tangent law of refraction formulated above, but, as first pointed out by Tan and Holland [13], this is alien to Snell's law because in geometrical optics a ray always bends toward the gradient of the index of refraction [6].

In an opposite case, the thermal resistivity can be an exponentially decreasing function of x

$$\rho(x)=\rho^0\exp(-\gamma x) \tag{56}$$

the corresponding fomulae follow from the previous ones when γ is replaced by $-\gamma$. (53) and (54) take respectively the form

$$y=y^0-\frac{c}{\rho^0}\int_{x^0}^{x}\frac{\mathrm{d}x'}{\exp(-\gamma x')}=\frac{c}{\gamma\rho^0}\left(\exp(\gamma x)-1\right) \tag{57}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{c}{\rho^0}\exp(\gamma x). \tag{58}$$

The slope of the heat ray increases exponentially with x, bending away from the direction of the initial slope $(\mathrm{d}y/\mathrm{d}x)^0=c/\rho^0$. This is in agreement with the tangent law of refraction from a denser to a rarer medium. In this case there is no asymptotic value of y. For the vanishing initial slope c/ρ^0 we obtain $y=0$ and $\mathrm{d}y/\mathrm{d}x=0$ for all x which means that a thermal ray initially in the direction of the resistivity gradient propagates undeviated. Otherwise, considering the inverted form of Eq. (57) and (58) we conclude that if the initial slope is infinite $(\rho^0/c=0)$ then x and $\mathrm{d}x/\mathrm{d}y$ are zero for all y. This means that a ray perpendicular to the resistivity gradient (tangent to a surface of the constant resistivity) also propagates undeviated. Again, while this is consistent with the tangent law, it is alien to Snell's law, as in geometrical optics a ray always bends toward the gradient of the index of refraction. These properties hold quite generally for all potential flows, where the tangent component of the intensity (**E**, grad P, grad T^{-1}, etc.) is continuous across a surface of constant resistivity, $E_{1t}=E_{2t}$, whereas the normal component of the same intensity exhibits a jump resulting from different conductivities, $\sigma_1 E_{1n}=\sigma_2 E_{2n}$.

From Eq. (45) the Pontryagin's Hamiltonian of the problem satisfies the HJB equation

$$H = -p_x = \max_u \left\{ p_y u - A_0^{-1}\rho(x)\left(1+u^2\right) \right\}. \tag{59}$$

Using in this result Eqs. (46) and (47) with $u = \mathrm{d}y/\mathrm{d}x$ yields

$$\frac{\partial R}{\partial x} + 2A_0^{-1}\rho(x)u^2 - A_0^{-1}\rho(x)\left(1+u^2\right) = \frac{\partial R}{\partial x} + A_0^{-1}\rho(x)\left(u^2-1\right) = 0 \tag{60}$$

whence the extremum Hamiltonian in terms of the $\mathrm{d}y/\mathrm{d}x$ is

$$H = -p_x = A_0^{-1}\rho(x)\left(u^2-1\right) = A_0^{-1}\rho(x)\left(\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 1\right). \tag{61}$$

Since the specific resistance ρ and hence the process Lagrangian depend on x, this Hamiltonian is not constant along the extremal path. However, if ρ is independent of x the Hamiltonian becomes constant and the family of extremal paths is represented by the bundle of straight lines

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\left(1 + HA_0/\rho^0\right)^{1/2}. \tag{62}$$

Yet a first integral exists without the above restriction, since the Lagrangian is independent of y. This means that $p \equiv \partial R/\partial y$ is constant along the extremal path, that is, from Eq. (47)

$$\frac{\partial R}{\partial y} = 2A_0^{-1}\rho(x)\frac{\mathrm{d}y}{\mathrm{d}x} \equiv 2A_0^{-1}c. \tag{63}$$

As $\mathrm{d}y/\mathrm{d}x = \tan\alpha$ this is equivalent to the tangent law of refraction, Eq. (50),

$$\rho(x)\tan\alpha(x) = c, \tag{50}$$

Hence, the function of the optimal resistivity has the structure

$$R(x, y) = A_0^{-1}\left(2cy + f(x)\right). \tag{64}$$

From Eq. (47) the optimal deviation function $u = \mathrm{d}y/\mathrm{d}x$ is

$$u = \frac{A_0}{2\rho(x)}\frac{\partial R}{\partial y}. \tag{65}$$

Its substitution in Eq. (46)

$$\frac{\partial R}{\partial x} + \frac{\partial R}{\partial y}u - A_0^{-1}\rho(x)\left(1+u^2\right) = 0 \tag{46}$$

yields the Hamilton–Jacobi equation for the extremals of the problem

$$\frac{\partial R}{\partial x} + A_0^{-1}\rho(x)\left(\left(\frac{A_0}{2\rho(x)}\frac{\partial R}{\partial y}\right)^2 - 1\right) = 0. \tag{66}$$

The optimal resistivity function, Eq. (64), implies that $\rho R/\rho y = 2cA_0^{-1}$; with this result Eqs. (64) and (66) yield an equation for the unknown function $f(x)$

$$\frac{\partial R}{\partial x} + A_0^{-1}\rho(x)\left(\left(\frac{c}{\rho(x)}\right)^2 - 1\right) = A_0^{-1}\frac{\mathrm{d}f(x)}{\mathrm{d}x} + A_0^{-1}\rho(x)\left(\left(\frac{c}{\rho(x)}\right)^2 - 1\right) = 0 \tag{67}$$

which can be simplified to the form

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \rho(x) - \rho^{-1}(x)c^2. \tag{68}$$

Therefore the function f is obtained in the form of the integral

$$f(x) = \int_0^x \left[\rho(x') - \rho^{-1}(x')c^2\right]\mathrm{d}x' + f(x^0), \tag{69}$$

where $f(x^0) = 0$, in order to satisfy the initial condition $R(x^0, y^0) = 0$ for the extremal function (64). Hence the potential function describing the minimum resistance is

$$R(x, y) = A_0^{-1}\left(2cy + \int_0^x \left[\rho(x') - \rho^{-1}(x')c^2\right]\mathrm{d}x'\right). \tag{70}$$

For the exponential resistivity law, Eq. (52),

$$\rho(x) = \rho^0\exp(\gamma x) \tag{52}$$

the potential function $R(x, y)$ is

$$R(x, y) = A_0^{-1}\left(2cy + \int_0^x \left[\rho^0\exp\left(\gamma x'\right) - \left(\rho^0\right)^{-1}c^2\exp\left(-\gamma x'\right)\right]\mathrm{d}x'\right) = A_0^{-1}\left(2cy + \gamma^{-1}\rho^0\left[\exp(\gamma x) - 1\right] + \gamma^{-1}\left(\rho^0\right)^{-1}c^2\left[\exp(-\gamma x) - 1\right]\right). \tag{71}$$

When Eq. (56) holds, the above formula should contain $-\gamma$ in place of γ.

Let us verify if the obtained function R in Eq. (71), satisfies the HJB equation, Eq. (45), or the related Hamilton–Jacobi equation, Eq. (66). The first partial derivative, $p = \partial R/\partial y = 2cA_0^{-1}$, in agreement with Eqs. (47) and (63). The second derivative, $p_x = \partial R/\partial x$, is

$$\frac{\partial R}{\partial x} = A_0^{-1}\frac{\mathrm{d}f(x)}{\mathrm{d}x} = A_0^{-1}\rho(x) - A_0^{-1}\rho^{-1}(x)c^2$$

$$= A_0^{-1}\rho^0\exp(\gamma x) - A_0^{-1}\left(\rho^0\right)^{-1}(-\gamma x)c^2. \tag{72}$$

Substituting these derivatives into the left-hand side of the Hamilton–Jacobi equation for our problem, Eq. (66), yields zero as

$$\mathscr{L}_{\mathrm{HB}} \equiv A_0^{-1}\rho^0\exp(\gamma x) - A_0^{-1}\left(\rho^0\right)^{-1}\exp(-\gamma x)c^2 + A_0^{-1}\rho^0\exp(\gamma x)\left(\left(\frac{c}{\rho^0\exp(\gamma x)}\right)^2 - 1\right) = 0. \tag{73}$$

Thus the Hamilton–Jacobi equation is satisfied. Furthermore, substituting these derivatives into the left hand side of the HJB equation of the problem, Eq. (45), yields an expression

$$\mathscr{L}_{\mathrm{HBJ}} \equiv A_0^{-1}\rho^0\exp(\gamma x) - A_0^{-1}\left(\rho^0\right)^{-1}\exp(-\gamma x)c^2 + \max_u\left\{2cA_0^{-1}u - A_0^{-1}\rho^0\exp(\gamma x)\left(1+u^2\right)\right\}. \tag{74}$$

The maximum in the above expression corresponds to u satisfying the equation

$$u = c\rho^{-1}(x) = c\left(\rho^0\right)^{-1}\exp(-\gamma x) \tag{75}$$

in agreement with Eqs. (50), (52) and (63). The second derivative of the maximized expression with respect to u is negative, hence the maximum occurs. Using the extremal u, Eq. (75), in Eq. (74) yields

$$L_{\mathrm{HBJ}} \equiv A_0^{-1}\rho^0\exp(\gamma x) - A_0^{-1}\left(\rho^0\right)^{-1}\exp(-\gamma x)c^2 + 2c^2A_0^{-1}\left(\rho^0\right)^{-1}\exp(-\gamma x) - A_0^{-1}\rho^0\exp(\gamma x) - A_0^{-1}\rho^0\exp(\gamma x)\left[c\left(\rho^0\right)^{-1}\exp(-\gamma x)\right]^2 = 0. \tag{76}$$

This proves that the obtained function $R(x, y)$, Eq. (71), satisfies the HJB equation of the problem, Eq. (45). See Rund [9] for a comprehensive discussion of connection between the variational calculus and Hamilton–Jacobi theory.

7. Summary of results for continuous problem of thermal rays

Before passing to the discrete formulation we will recapitulate results obtained until now putting them sometimes in a suitable modified form. The theory of traveling thermal waves involves an optimal control problem for the minimum of the resistivity integral

$$W^N = \int_{t_1}^{t_2} A_0^{-1}\rho(x)\left(1+u^2\right)\mathrm{d}x \tag{42}$$

subject to the control u defined by the state equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u \tag{43}$$

The minimal resistivity function of the continuous problem defined as

$$R\left(x^i, y^i, x^f, u^f\right) \equiv \min\int_{t_1}^{t_2} A_0^{-1}\rho(x)\left(1+u^2\right)\mathrm{d}x \tag{44}$$

satisfies the HJB equation

$$\frac{\partial R}{\partial x} + \max_u\left\{\frac{\partial R}{\partial y}u - A_0^{-1}\rho(x)\left(1+u^2\right)\right\} = 0. \tag{45}$$

Extremizing the Hamiltonian in the above HJB equation yields as an optimal control

$$u = \frac{A_0}{2\rho(x)}\frac{\partial R}{\partial y}. \tag{65}$$

This optimality condition can be written in the form of the tangent law of bending for a thermal ray

$$\rho(x)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}A_0\frac{\partial R}{\partial y} \equiv c, \tag{50}$$

where c may be both positive or negative constant. The constancy of the partial derivative $\partial R/\partial y$ follows from an explicit independence of the model Lagrangian with respect to y. A suitable integral formula for the bending constant in terms of the deviation $y - y^0$ is

$$c = \left(y - y^0\right)\left(\int_{x^0}^{x}\rho^{-1}(x')\,\mathrm{d}x'\right)^{-1}. \tag{51}$$

Expressing the optimal control u in the HJB equation in terms of $p = \partial R/\partial y$ yields the Hamilton–Jacobi equation for the continuous problem

$$\frac{\partial R}{\partial x} + A_0^{-1}\rho(x)\left(\left(\frac{A_0}{2\rho(x)}\frac{\partial R}{\partial y}\right)^2 - 1\right) = 0 \tag{66}$$

where the second term of the left-hand side expression is the optimal Hamiltonian. The solution to this equation can always be broken down to quadratures. Using the separation of variables the potential function describing the minimum resistance follows in the form

$$R(x, y, c) = A_0^{-1}\left(2c\left(y - y^0\right) + \int_{x^0}^{x}\left\lfloor\rho(x') - \rho^{-1}(x')c^2\right\rceil\mathrm{d}x'\right). \tag{70}$$

It may be verified that the above function satisfies both HJB equation (45) and Hamilton–Jacobi equation (66). However if the function of specific resistivity $\rho(x)$ is too complicated the resulting integrals cannot be evaluated explicitly. Hence, the important role of the discrete approach using Bellman's recurrence equation to get a numerical solution. For this purpose a discrete extension of Pontryagin's theory is used, which differs from standard discrete approaches [3] due to freedom in choice of the intervals of the state variables at a given number of stages [10,11].

8. A discrete approach using Bellman's equation or a stage criterion

We cast the problem into the one by minimizing the discrete function

$$W^N \equiv \sum_1^N A_0^{-1}\rho(x^n)\big(1+(u^n)^2\big)\theta^n \tag{77}$$

subject to the discrete constraints

$$\frac{y^n - y^{n-1}}{\theta^n} = u^n \tag{78}$$

and

$$\frac{x^n - x^{n-1}}{\theta^n} = 1. \tag{79}$$

The minimum resistivity function for this discrete problem is defined as

$$R^n(y^n, x^n) \equiv \min \sum_1^n A_0^{-1}\rho(x^n)\big(1+(u^n)^2\big)\theta^n. \tag{80}$$

The discrete Hamiltonian function is

$$H^{n-1} = \frac{\partial R^{n-1}}{\partial y^{n-1}}u^n - A_0^{-1}\rho(x^n)\big(1+(u^n)^2\big). \tag{81}$$

As implied by the discrete HJB equation of the problem,

$$\max_{u^n}\left\{\frac{\partial R^{n-1}}{\partial x^{n-1}} + \frac{\partial R^{n-1}}{\partial y^{n-1}}u^n - A_0^{-1}\rho(x^n)\big(1+(u^n)^2\big)\right\} = 0 \tag{82}$$

in an optimal process the Hamiltonian function has a maximum with respect to $u = \mathrm{d}y/\mathrm{d}x$. However, following our philosophy, instead of solving the discrete HJB equation or a related discrete Hamilton–Jacobi equation of the problem (see Eq. (89)), in our approach we solve more basic discrete equations: Bellman's recurrence equation or a recurrence equation for the so-called stage criterion, as those ones for which the numerical procedures are the most efficient.

Bellman's recurrence equation which deals with the above minimum resistivity function has the form

$$\begin{aligned} R^n(y^n, x^n) = \min_{u^n,\,\theta^n}\Big\{&A_0^{-1}\rho(x^n)\big(1+(u^n)^2\big)\theta^n \\ &+ R^{n-1}\big(y^n - u^n\theta^n, x^n - \theta^n\big)\Big\} = 0. \end{aligned} \tag{83}$$

in agreement with general recurrence equations of this type {e.g. Eq. (84) in [12]}. This is a suitable functional equation which should be solved numerically whenever the specific resistivity function $\rho(x^n)$ is too complicated to solve the problem analytically. Yet, still more general equation

$$\begin{aligned} \max_{u^n,\,\theta^n,\,x^n,\,y^n}\Big\{&R^n\big(y^n, x^n\big) - R^{n-1}\big(y^n - u^n\theta^n, x^n - \theta^n\big) \\ &- A_0^{-1}\rho(x^n)\big(1+(u^n)^2\big)\theta^n\Big\} = 0. \end{aligned} \tag{84}$$

describing the so-called stage criterion can be used for the purpose of prior evaluation of the properties of partial derivatives of R^n, in order to exploit these properties in further considerations. We note that:

- From the stage criterion Bellman's recurrence equation is recovered at fixed x^n and y^n. On the other hand, when variations of the state variables are admitted discrete characteristic sets are obtained which govern the partial derivatives $\partial R^{n-1}/\partial x^{n-1}$ and $\partial R^{n-1}/\partial y^{n-1}$.
- The stationary condition for Eq. (84) with respect to the intervals θ^n yields the expression which describes vanishing 'enlarged Hamiltonian'

$$\frac{\partial R^{n-1}}{\partial x^{n-1}} + \frac{\partial R^{n-1}}{\partial y^{n-1}}u^n - A_0^{-1}\rho(x^n)\big(1+(u^n)^2\big) = 0, \tag{85}$$

where the energy-type Hamiltonian appears

$$H^{n-1} = \frac{\partial R^{n-1}}{\partial y^{n-1}}u^n - A_0^{-1}\rho(x^n)\big(1+(u^n)^2\big). \tag{86}$$

- The discrete HJB equation of the problem

$$\frac{\partial R^{n-1}}{\partial x^{n-1}} + \max_{u^n}\left\{\frac{\partial R^{n-1}}{\partial y^{n-1}}u^n - A_0^{-1}\rho(x^n)\big(1+(u^n)^2\big)\right\} = 0 \tag{87}$$

in which the partial derivatives $\partial R^{n-1}/\partial x^{n-1}$ and $\partial R^{n-1}/\partial y^{n-1}$ are fixed, yields the optimality condition for the Hamiltonians with respect to u^n

$$\frac{\partial H^{n-1}}{\partial u^n} = \frac{\partial R^{n-1}}{\partial y^{n-1}} - 2A_0^{-1}\rho(x^n)u^n = 0. \tag{88}$$

This is the same optimality condition as that implied by the 'Caratheodory type' stage criterion, Eq. (84). An equation of Hamilton–Jacobi type follows from Eqs. (87) and (88) as a discrete extension of Eq. (66)

$$\frac{\partial R^{n-1}}{\partial x^{n-1}} + A_0^{-1}\rho(x^n)\left(\frac{1}{4}\left(\frac{\partial R^{n-1}}{\partial y^{n-1}}\right)^2 A_0^2\rho^{-2}(x^n) - 1\right) = 0. \tag{89}$$

In view of the x-dependent resistivity $\rho(x^n)$, the Hamiltonian is not constant along the discrete path. However, since the process Lagrangian is independent of y, the partial derivative $\partial R/\partial y$ is constant along the path. Indeed, the extremum condition for the stage criterion (84) with respect to y^n (the canonical equation for y^n in the phase space)

$$\frac{\partial R^n/\partial y^n - \partial R^{n-1}/\partial y^{n-1}}{\theta^n} = -\frac{\partial H^{n-1}}{\partial y^n} = 0 \quad \left(= \frac{p^n - p^{n-1}}{\theta^n} \right) \tag{90}$$

implies that in our case

$$\frac{\partial R^n}{\partial y^n} = \frac{\partial R^{n-1}}{\partial y^{n-1}} = \dots \frac{\partial R^1}{\partial y^1} = p. \tag{91}$$

Note that this condition restricts the functions R^n in the recurrence equation to be linear with respect to y^n, and allows the reduction of this equation to a one-dimensional relationship. The condition is not a representation of a tangent law of bending, but rather a condition for the optimality of subsequent states, when the controls u^n and θ^n are given. However, with the above optimum condition, the Hamiltonian can be written as

$$H^{n-1} = pu^n - A_0^{-1}\rho(x^n)\left(1 + (u^n)^2\right) = -A_0^{-1}\rho(x^n)\left(1 + (u^n)^2 - A_0\rho^{-1}(x^n)pu^n\right). \tag{92}$$

In terms of the bending constant $c = (1/2)A_0 p$ the Hamiltonian is

$$H^{n-1} = -A_0^{-1}\rho(x^n)\left(1 + (u^n)^2 - 2c\rho^{-1}(x^n)u^n\right). \tag{93}$$

The condition for maximum of the Hamiltonian with respect to u^n, Eq. (88) above, or

$$2A_0^{-1}\rho(x^n)u^n = 2A_0^{-1}\rho(x^n)\frac{y^n - y^{n-1}}{x^n - x^{n-1}} = p \tag{94}$$

describes the tangent law of refraction for a discrete thermal ray for $p = 2A_0^{-1}c$, so that

$$\rho(x^n)\tan\alpha(x^n) = c. \tag{95}$$

The property of constancy of $\partial R^n/\partial y^n$ allows us to reduce the state dimensionality by eliminating y from the set of the state variables. This is described below.

9. Reduced state space and role of transformed resistance potentials

We introduce the Lagrange multiplier $p = 2A_0^{-1}c$ associated with the constraint describing the total change of the variable y between the two given points (x^0, y^0) and (x^n, y^n)

$$\sum_1^n u^k\theta^k = y^n - y^0 \tag{96}$$

With the help of p, the problem can be transformed into the one with an unspecified final coordinate y^n, whose value results from an accepted value of the constant multiplier $p = 2A_0^{-1}c$. In other words, it is the freedom of the final coordinate y^n which yields the condition for the Lagrange multiplier. The transformed problem is described by the asterisk optimal function R_*^n, a new quantity defined by the set of equations

$$R_*^n(x^n, y^n, p) = R^n(x^n, y^n) - p(y^n - y^0), \tag{97}$$

$$\partial R_*^n/\partial x^n = \partial R^n/\partial x^n \tag{98}$$

and

$$0 = \partial R_*^n/\partial y^n = \partial R^n/\partial y^n - p. \tag{99}$$

These equations imply the link between R^* and R by the Legendre transformation

$$R_*^n = R^n - \left(\partial R^n/\partial y^n\right)\left(y^n - y^0\right) \tag{100}$$

$$R^n = R_*^n + p\left(y^n - y^0\right) = R_*^n - p\partial R_*^n/\partial p \tag{101}$$

Note that it is the deviation $y - y^0$ rather than the absolute coordinate y which appears in the Legendre transformation. We assume that these relations are satisfied for an arbitrary n. In view of the constraint, Eq. (96), the transformed problem is governed by the recurrence equation for the 'stage criterion' [2,11,12].

$$\max_{u^n,\,\theta^n,\,x^n}\left\{R_*^n(x^n, p) - R_*^{n-1}(x^n - \theta^n, p) - A_0^{-1}\rho(x^n)\left(1 + (u^n)^2 - pA_0\rho^{-1}(x^n)u^n\right)\theta^n\right\} = 0 \tag{102}$$

or by Bellman's recurrence equation

$$R_*^n(x^n, p) = \min_{u^n,\,\theta^n}\left\{A_0^{-1}\rho(x^n)\left(1 + (u^n)^2 - pA_0\rho^{-1}(x^n)u^n\right)\theta^n + R_*^{n-1}(x^n - \theta^n, p)\right\}. \tag{103}$$

which can also be investigated in an alternative form with the control x^{n-1} instead of θ^n

$$R_*^n(x^n, p) = \min_{u^n,\,\theta^n}\left\{A_0^{-1}\rho(x^n) \times \left(1 + (u^n)^2 - pA_0\rho^{-1}(x^n)u^n\right)(x^n - x^{n-1}) + R_*^{n-1}(x^{n-1}, p)\right\}. \tag{104}$$

Still another form uses as the parameter the integration constant of the bending law, c,

$$R_*^n(x^n, c) = \min_{u^n,\,\theta^n} \Big\{ A_0^{-1}\rho(x^n)\Big(1+(u^n)^2 - 2c\rho^{-1}(x^n)u^n\Big)\theta^n + R_*^{-1}(x^n-\theta^n, c)\Big\}. \qquad (105)$$

($p = 2A_0^{-1}c$). In view of the vanishing derivative $\partial R_*^n/\partial y^n$, the asterisk Hamiltonian coincides with the original one

$$H_*^{n-1} = \frac{\partial R^{n-1}}{\partial y_*^{n-1}}u^n - A_0^{-1}\rho(x^n)\Big(1+(u^n)^2 - pA_0\rho^{-1}(x^n)u^n\Big) = -A_0^{-1}\rho(x^n)\Big(1+(u^n)^2 - pA_0\rho^{-1}(x^n)u^n\Big) = H^{n-1} \qquad (106)$$

We also verify that the condition for the free extremal control u^n, $\partial H_*^{n-1}/\partial u^n = 0$, along with the incorporated condition for the free final coordinate y^n yield the bending condition, Eq. (94) above.

Let us discuss the solution of Bellman's recurrence equation using the algorithm in Eq. (104). Assuming a given starting point x^0 and y^0, not necessarily the point (0, 0), we shall see explicitly the role of the deviation $y - y^0$. We find for $n = 1$

$$R_*^1(x^1, p) \min_{u^1,\,\theta^1=\Delta x^1} \Big\{ A_0^{-1}\rho(x^1) \times \Big(1+(u^1)^2 - pA_0\rho^{-1}(x^n)u^1\Big)(x^1 - x^0)\Big\}. \qquad (107)$$

This yields the stationary condition for the optimal control u^1 in the form

$$2A_0^{-1}\rho(x^1)u^1 = p \qquad (108)$$

which is precisely both the condition for the optimal control, Eq. (94) for $n = 1$, and the c-representation of the bending law for

$$p = 2A_0^{-1}c. \qquad (109)$$

Using the optimal control

$$u^1 = \frac{1}{2}pA_0\rho^{-1}(x^1) \qquad (110)$$

in Eq. (107), the function $R_*^1(x^1, p)$ is obtained as

$$R_*^1(x^1, p) = \Big(A_0^{-1}\rho(x^1) - \frac{1}{4}p^2A_0\rho^{-1}(x^1)\Big)(x^1 - x^0). \qquad (111)$$

The free coordinate y^1 follows from Eq. (110) and the definition of u in terms of the multiplier p as

$$y^1 = y^0 + \frac{1}{2}A_0\rho^{-1}(x^1)p(x^1 - x^0) \qquad (112)$$

This is in agreement with $y^n - y^0 = -\partial R_*^n/\partial p$ or (when $y^0 = 0$) $y^n = -\partial R_*^n/\partial p$ for each n.

The original function R^1 is obtained as the Legendre transform of R_*^n, which deals with the deviation $y - y^0$ rather than with an absolute value of y

$$R^1 = R_*^1 + p\big(y^1 - y^0\big) = R_*^1 - p\partial R_*^1/\partial p \qquad (113)$$

In terms of the variables x^1 and p the procedure yields

$$\begin{aligned} R^1\big(x^1, y^1\big) &= \Big(A_0^{-1}\rho(x^1) - \frac{1}{4}p^2A_0\rho^{-1}(x^1)\Big)(x^1 - x^0) - p\partial R_*^1/\partial p \\ &= \Big(A_0^{-1}\rho(x^1) - \frac{1}{4}p^2A_0\rho^{-1}(x^1)\Big)(x^1 - x^0) + \frac{1}{2}A_0\rho^{-1}(x^1)p^2(x^1 - x^0) \\ &= \Big(A_0^{-1}\rho(x^1) + \frac{1}{4}p^2A_0\rho^{-1}(x^1)\Big)(x^1 - x^0). \end{aligned} \qquad (114)$$

This corresponds to the deviation

$$y^1 - y^0 = -\partial R_*^1/\partial p = \frac{1}{2}A_0\rho^{-1}(x^1)p(x^1 - x^0) \qquad (115)$$

and the representation of p in terms of y^1

$$p = 2A_0^{-1}\rho(x^1)\frac{y^1 - y^0}{x^1 - x^0}. \qquad (116)$$

Hence the function R^1 follows in terms of x^1 and y^1 as

$$\begin{aligned} R^1\big(x^1, y^1\big) &= \Big(A_0^{-1}\rho(x^1) + \frac{1}{4}p^2A_0\rho^{-1}(x^1)\Big)(x^1 - x^0) \\ &= \left(A_0^{-1}\rho(x^1) + \frac{1}{4}\left(2A_0^{-1}\rho(x^1)\frac{y^1 - y^0}{x^1 - x^0}\right)^2 \times A_0\rho^{-1}(x^1)\right)(x^1 - x^0) \end{aligned} \qquad (117)$$

which can be simplified into the form

$$R^1\big(x^1, y^1\big) = A_0^{-1}\rho(x^1)\left\{1 + \left(\frac{y^1 - y^0}{x^1 - x^0}\right)^2\right\}(x^1 - x^0). \qquad (118)$$

For $n = 2$ we find the function $R_*^2(x^2, p)$ from the general recurrence Eq. (103) as

$$R_*^2(x^2, p) = \min_{u^2,\,\theta^2} \Big\{ A_0^{-1}\rho(x^2)\Big(1+(u^2)^2 - pA_0\rho^{-1}(x^2)u^2\Big)\theta^2 + \Big(A_0^{-1}\rho\big(x^2-\theta^2\big) - \frac{1}{4}p^2A_0\rho^{-1}\big(x^2-\theta^2\big)\Big) \times \big(x^2 - \theta^2 - x^0\big)\Big\}. \qquad (119)$$

or, in terms of x^1 as an alternative control replacing θ^2, as in Eq. (104),

$$R_*^2(x^2,p)=\min_{u^2,\,x^1}\left\{A_0^{-1}\rho(x^2)\left(1+(u^2)^2-pA_0\rho^{-1}(x^1)u^2\right)\times(x^2-x^1)+\left(A_0^{-1}\rho(x^1)-\frac{1}{4}p^2A_0\rho^{-1}(x^1)\right)\times(x^1-x^0)\right\} \tag{120}$$

We note that for each of these equations the extremum condition with respect to u^2 takes the form of that obtained for $n=1$, but the indices correspond now with $n=2$

$$2A_0^{-1}\rho(x^2)u^2=p\quad\left(=2A_0^{-1}c\right). \tag{121}$$

Substituting the resulting optimal control

$$u^2=\frac{1}{2}pA_0\rho^{-1}(x^2) \tag{122}$$

into Eq. (120), the function $R_*^2(x^2,p)$ follows from the condition

$$R_*^2(x^2,p)=\min_{x^1}\left\{\left(A_0^{-1}\rho(x^2)-\frac{1}{4}p^2A_0\rho^{-1}(x^2)\right)\times(x^2-x^1)+\left(A_0^{-1}\rho(x^1)-\frac{1}{4}p^2A_0\rho^{-1}(x^1)\right)(x^1-x^0)\right\} \tag{123}$$

in which extremizing is with respect to the intermediate state x^1 only, for fixed states x^0 and x^2. The condition for optimality of x^1 yields an equation

$$\left(A_0^{-1}\rho(x^1)-\frac{1}{4}p^2A_0\rho^{-1}(x^1)\right)-\left(A_0^{-1}\rho(x^2)-\frac{1}{4}p^2A_0\rho^{-1}(x^2)\right)+\left(A_0^{-1}\,\mathrm{d}\rho(x^1)/\mathrm{d}x^1-\frac{1}{4}p^2A_0\,\mathrm{d}\rho^{-1}(x^1)/\mathrm{d}x^1\right)\times(x^1-x^0)=0. \tag{124}$$

which is characteristic of only the discrete processes. It defines the optimal interstage coordinate x^1 in terms of given coordinates x^0 and x^2, and for an arbitrary specific resistivity it should be solved numerically. Alternatively a direct minimizing of the braces expression in Eq. (123) with respect to x^1 should be performed.

Summing up the above results we conclude that, with the bending law incorporated, the procedure finding the sequence of optimal functions R_*^n is broken down to the optimization of the sum

$$R_*^n(x^n,p)=\min_{\{x^{k-1}\}}\sum_{k=1}^{n}\left\{\left(A_0^{-1}\rho(x^k)-\frac{1}{4}p^2A_0\rho^{-1}(x^k)\right)(x^k-x^{k-1})\right\} \tag{125}$$

with the interstage states x^{k-1} as only controls. The final recurrence equation is

$$R_*^n(x^n,p)=\min_{x^{n-1}}\left\{\left(A_0^{-1}\rho(x^n)-\frac{1}{4}p^2A_0\rho^{-1}(x^n)\right)\times(x^n-x^{n-1})+R_*^{n-1}(x^{n-1},p)\right\} \tag{126}$$

This equation should be applied for $n=2,3\ldots N$ using the one-stage function defined as

$$R_*^1(x^1,p)=\left(A_0^{-1}\rho(x^1)-\frac{1}{4}p^2A_0\rho^{-1}(x^1)\right)(x^1-x^0). \tag{127}$$

The optimal results for the next stages are obtained numerically with the help of Eqs. (126) and (127). The optimal interstage coordinate x^{n-1} satisfies an equation

$$A_0^{-1}\rho(x^n)-\frac{1}{4}p^2A_0\rho^{-1}(x^n)=\frac{\mathrm{d}R^{n-1}(x^{n-1},p)}{\mathrm{d}x^{n-1}}. \tag{128}$$

from which tabular data $x^{n-1}(x^n)$ can be obtained via a numerical procedure. Most often, however, a direct minimizing of the braces expression in Eq. (126) with respect to x^{n-1} is performed.

10. Limiting transition to continuous potentials

In a continuous process, the limiting function $R_*(x,p)$ is an integral

$$R_*(x,p)=\int_{x^0}^{x}\left(A_0^{-1}\rho(x')-\frac{1}{4}p^2A_0\rho^{-1}(x')\right)\mathrm{d}x'. \tag{129}$$

Its potential property is obvious, even for variable specific resistivity.

The original resistivity potential (without asterisk) is the Legendre transform of the above function with respect to p. In terms of the variables p and x

$$R(x,p)=R_*+p(y-y^0)=R_*-p\partial R_*/\partial p=\int_{x^0}^{x}A_0^{-1}\rho(x')\,\mathrm{d}x'+\frac{1}{4}p^2\int_{x^0}^{x}A_0\rho^{-1}(x')\,\mathrm{d}x' \tag{130}$$

or, in terms of c and x,

$$R(x,c)=A_0^{-1}\left\{\int_{x^0}^{x}\rho(x')\,\mathrm{d}x'+c^2\int_{x^0}^{x}\rho^{-1}(x')\,\mathrm{d}x'\right\}. \tag{131}$$

Using in the above integral the relationship

$$y-y^0=-\partial R_*/\partial p=\int_{x^0}^{x}\frac{1}{2}pA_0\rho^{-1}(x')\,\mathrm{d}x' \tag{132}$$

in the form

$$p=2A_0^{-1}\big(y-y^0\big)\bigg(\int_{x^0}^{x}\rho^{-1}(x')\,\mathrm{d}x'\bigg)^{-1}\quad\big(=2A_0^{-1}c\big) \tag{133}$$

the original function $R(x, y)$ is obtained as

$$\begin{aligned}R(x,y)&=\int_{x^0}^{x}A_0^{-1}\rho(x')\,\mathrm{d}x'+\frac{1}{4}\Big[2A_0^{-1}\big(y-y^0\big)\Big]^2\\&\quad\times\bigg(\int_{x^0}^{x}\rho^{-1}(x')\,\mathrm{d}x'\bigg)^{-2}\int_{x^0}^{x}A_0\rho^{-1}(x')\,\mathrm{d}x'\\&=\int_{x^0}^{x}A_0^{-1}\rho(x')\,\mathrm{d}x'+A_0^{-1}\big(y-y^0\big)^2\\&\quad\times\bigg(\int_{x^0}^{x}\rho^{-1}(x')\,\mathrm{d}x'\bigg)^{-1}\end{aligned} \tag{134}$$

This function constitutes the solution to the continuous Hamilton–Jacobi equation, Eq. (66). The present form of $R(x, y)$ is the most useful as it does not contain the bending constant c or the related constant p. Note that each numerical solution to recurrence Eq. (83) for finite number of stages n represents a finite-stage generalization of the solution (134); this numerical solution automatically accomplishes the numerical integration contained in Eq. (134).

Yet an equivalent form, referred to a 'mixed representation', can be obtained

$$\begin{aligned}R(x,y,p)&=\int_{x^0}^{x}A_0^{-1}\rho(x')\,\mathrm{d}x'+\frac{1}{4}p\big(y-y^0\big)\\&=A_0^{-1}\bigg\{\int_{x^0}^{x}\rho(x')\,\mathrm{d}x'+\frac{1}{2}c\big(y-y^0\big)\bigg\}\\&=R(x,y,c)\end{aligned} \tag{135}$$

Equivalence of dynamic programming solutions, Eqs. (130) and (134) or (135), (70) obtained earlier by the method of separation of variables should be shown. Indeed, for $c=(1/2)A_0p$, Eq. (70) agrees with the Legendre transformation of Eq. (129) contained in Eq. (130)

$$\begin{aligned}R(x,y,c)&=A_0^{-1}\bigg(2c\big(y-y^0\big)+\int_{x^0}^{x}\Big[\rho(x')\\&\qquad-\rho^{-1}(x')c^2\Big]\,\mathrm{d}x'\bigg)\\&=p\big(y-y^0\big)+A_0^{-1}\int_{x^0}^{x}\rho(x')\,\mathrm{d}x'\\&\quad-\frac{1}{4}A_0p^2\int_{x^0}^{x}\rho^{-1}(x')\,\mathrm{d}x'\\&=\int_{x^0}^{x}A_0^{-1}\rho(x')\,\mathrm{d}x'\\&\quad+\frac{1}{4}p^2\int_{x^0}^{x}A_0\rho^{-1}(x')\,\mathrm{d}x'\\&=R(x,p).\end{aligned} \tag{136}$$

Let us consider now a special case when the specific resistivity is independent of x. The conditions (124) and (128) are then satisfied for an arbitrary x^1 and x^{n-1} which means that the discrete decision problem degenerates in this case. This is because the extremals are then straight lines and the magnitude of intervals θ^n does not influence the value of the braces expression in Eq. (123) above. This conclusion is valid for larger n as well. As it follows from Eq. (123), the function $R_*^2(x^2, p)$ equals in this special case

$$R_*^2\big(x^2,p\big)=\bigg(A_0^{-1}\rho-\frac{1}{4}p^2A_0\rho^{-1}\bigg)\big(x^2-x^0\big) \tag{137}$$

and we can easily find the general result

$$R_*^n\big(x^n,p\big)=\bigg(A_0^{-1}\rho-\frac{1}{4}p^2A_0\rho^{-1}\bigg)\big(x^n-x^0\big) \tag{138}$$

which holds only for a constant specific resistivity ρ. Its continuous limit is

$$R_*(x,p)=\bigg(A_0^{-1}\rho-\frac{1}{4}p^2A_0\rho^{-1}\bigg)\big(x-x^0\big) \tag{139}$$

This corresponds to thermal rays as straight lines. The original resistivity function is the Legendre transform of the asterisk function. In terms of the variables p and x

$$\begin{aligned}R^n\big(x^n,p\big)&=R_*^n+p\big(y^n-y^0\big)=R_*^n-p\partial R_*^n/\partial p\\&=\bigg(A_0^{-1}\rho+\frac{1}{4}p^2A_0\rho^{-1}\bigg)\big(x^n-x^0\big).\end{aligned} \tag{140}$$

We can now use an integral form of Eq. (132)

$$y^n-y^0=\partial R_*^n/\partial p=\frac{1}{2}A_0\rho^{-1}p\big(x^n-x^0\big) \tag{141}$$

to obtain

$$p = 2A_0^{-1}\rho\frac{y^n - y^0}{x^n - x^0}. \tag{142}$$

After substituting the above result into Eq. (140), the original function $R^n(x^n, y^n)$ is obtained in terms of x^n and y^n as

$$R^n\left(x^n, y^n\right) = A_0^{-1}\rho\left\{1 + \left(\frac{y^n - y^0}{x^n - x^0}\right)^2\right\}\left(x^n - x^0\right). \tag{143}$$

In the limiting continuous process, the constant resistivity solution is represented by

$$R(x, y) = A_0^{-1}\rho\left\{1 + \left(\frac{y - y^0}{x - x^0}\right)^2\right\}\left(x - x^0\right). \tag{144}$$

This corresponds to the bundle of rectilinear rays with various constant $u = \mathrm{d}y/\mathrm{d}x$.

Application of this theory to composites should be underlined, especially when they are designed and θ^n. Yet, for ready or existing composites θ^n are constrained. The corresponding theory can be found in [12].

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